BBB-003-1164002 Seat No. ________

M. Sc. (Sem. IV) (CBCS) Examination

July - 2021

Mathematics : CMT - 4002

(Integration Theory)

Faculty Code : 003
Subject Code : 1164002

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions:
(1) Answer any **five** questions.
(2) Each question carries **14** marks.
(3) There are **10** questions.

1 Answer the following *seven* questions: (7 X 2 = 14) [14]

(1) Define: σ −algebra of subsets of a nonempty set X. Also give an example of an algebra which is not a σ −algebra.

(2) If $\mathcal{A}$ is a σ −algebra of subsets of X and $\mathcal{A} \neq \phi$ then prove that $\phi, X \in \mathcal{A}$.

(3) Define: Atomic Measure.

(4) Give only statement of Lebesgue Decomposition Theorem.

(5) Define: Positive set, negative set and null set.

(6) Prove that, every compact subset of K of a Hausdorff space is closed.

(7) Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \gamma)$ be complete measure spaces. Then prove that, $\mathcal{R} = \{A \times B \subset X \times Y \ / \ A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$ is a semi-algebra.

2 Answer the following *seven* questions: (7 X 2 = 14) [14]

(1) Define: Measure of a measurable space with example.

(2) Define: Signed Measure.

(3) Give the statement of Radon – Nikodym Theorem for signed measure.

(4) Give the statement of Tonelli's Theorem.

(5) Define: Mutually singular measures with example.

(6) Let X be a locally compact T_2 −space and K be a compact G_δ −set in X then prove that, $K \in B_a(X)$.

(7) Prove that, a function is continuous if and only if it is lower semi continuous as well as upper semi continuous.

3 Answer the following *two* questions: (2 X 7 = 14) [14]

a) Define: Algebra of subsets of a set X. If X is any set, prove that, $\mu: P(X) \longrightarrow [0, \infty]$ defined by

$$\mu(A) = \begin{cases} \text{the number of elements} & ; \text{if A is finite} \\ \infty & ; \text{if A is infinite} \end{cases}$$

is measure on $(X, P(X))$.

b) State and prove: Hahn Decomposition Theorem.

4 **Answer the following *two* questions: (2 X 7 = 14)** **[14]**

1. Define: Measure absolutely continuous with respect to another measure and mutually singular measures. If $(X, \mathcal{A})$ is a measurable space and γ, μ are signed measures on $(X, \mathcal{A})$, $\gamma \perp \mu$, $\gamma \ll \mu$ then prove that, $\gamma = 0$.
2. Let γ be a signed measure on $(X, \mathcal{A})$. Then prove that, $\exists$ unique measures γ^+ and γ^- on $(X, \mathcal{A})$ such that $\gamma = \gamma^+ - \gamma^-$ on $\mathcal{A}$, $\gamma^+ \perp \gamma^-$, where γ^+ and γ^- are positive and negative part of γ respectively.

5 **Answer the following *two* questions: (2 X 7 = 14)** **[14]**

1. Let $(X, \mathcal{A}, \mu)$ be a finite complete measure space, p, q be extended non-negative real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, g be integrable on $(X, \mathcal{A}, \mu)$ and $\left|\int_X g \, \phi \, d\mu\right| \leq M \cdot ||\phi||_p$, for all simple measurable function ϕ on X for some $M > 0$. Prove that, $g \in L^q(\mu)$.
2. If μ^* is an outer measure on a set X and $B = \{E \subseteq X / E \text{ is } \mu^* - \text{measurable}\}$. Prove that, B is σ −algebra of subsets of X.

6 **Answer the following *two* questions: (2 X 7 = 14)** **[14]**

1. Define: Baire measure on the real line. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be monotonically increasing and continuous function on the right. Prove that, $\exists$ a baire measure μ on the real line such that $\mu(a, b] = f(a) - f(b), \forall a, b \in \mathbb{R}$ and $a < b$.
2. If X is a countable set and μ is the counting measure on $\big(X, P(X)\big)$. Prove that, $L^p(\mu) \cong l^p$, $1 \leq p \leq \infty$.

7 **Answer the following *two* questions: (2 X 7 = 14)** **[14]**

1. Let $(X, \mathcal{A})$ be a measurable space, $\mathcal{D} \subseteq \mathbb{R}$ be dense and B_α, $\alpha \in \mathcal{D}$ be measurable in $(X, \mathcal{A})$ such that $B_\alpha \subseteq B_\beta$, $\forall \alpha, \beta \in \mathcal{D}$ such that $\alpha < \beta$. Prove that, $\exists$ a unique measurable function $f: X \longrightarrow [-\infty, \infty]$ such that $f(x) \leq \alpha$, $\forall\, x \in B_\alpha$ and $f(x) \geq \alpha, \forall x \in X - B_\alpha$.
2. Let $\mathcal{m}$ be the σ −algebra generated by all lebesgue measurable subsets of $\mathbb{R}$ and μ be the lebesgue measure on $(\mathbb{R}, \mathcal{m})$. Prove that, μ is regular.

8 **Answer the following *two* questions: (2 X 7 = 14)** **[14]**

1. Let X be a topological space. Prove that,
   a) For $F \subseteq X$, $\chi_F: X \longrightarrow \{0,1\}$ is upper semi continuous if and only if F is closed in X.
   b) If $f_\alpha: X \longrightarrow \{0,1\}$ are upper semi continuous, $\forall \alpha \in \Lambda$ then prove that, $\inf_{\alpha \in \Lambda} f_\alpha$ is also upper semi continuous on X.
2. Let X be a locally compact T_2 −space. Then prove that, $B_a(X)$ is the σ −algebra generated by all compact G_δ −sets in X.

9 Answer the following *one* questions: (1 X 14 = 14) [14]

1. State and prove, Fubini's theorem.

10 Answer the following *one* questions: (1 X 14 = 14) **[14]**

1. Let X be a locally compact T_2 −space and $E \in B_a(X)$. Then prove that, either E is σ −bounded or $X - E$ is a σ −bounded set in X. Also prove that, E and $X - E$ both are σ −bounded then X must be σ −compact.
